Possible evidences from $H(z)$ parameter data for physics beyond ΛCDM

Rong-Jia Yang *[1,2,3,4,†]

[1] *College of Physics Science and Technology, Hebei University, Baoding 071002, China*
[2] *Hebei Key Lab of Optic-Electronic Information and Materials, Hebei University, Baoding 071002, China*
[3] *National-Local Joint Engineering Laboratory of New Energy Photoelectric Devices, Hebei University, Baoding 071002, China*
[4] *Key Laboratory of High-pricision Computation and Application of Quantum Field Theory of Hebei Province, Hebei University, Baoding 071002, China*

We analyse $H(z)$ parameter data with some conditions by using Lagrange mean value theorem in Calculus. We find that: (1) there exists at least one decelerated phase at 1 σ confidence level in the redshift range $(0.38, 0.59)$; (2) the equation of motion of dark energy may be less than -1 at 1 σ confidence level at some redshifts in the redshift range $(1.3, 1.53)$; (3) there exists at least one accelerated phase at 1 σ confidence level in the redshift range $(1.037, 1.944)$. These results may provide possible evidences for physics beyond ΛCDM.

PACS numbers:

I. INTRODUCTION

A great number of independent cosmological observations, such as supernova Ia (SNIa) at high redshift [1, 2], large-scale structure [3], and the cosmic microwave background anisotropy [4, 5], have confirmed that the Universe is experiencing an accelerated expansion. In order to explain this phenomenon, an unknown energy component (dubbed as dark energy) usually have to be introduced in the framework of general relativity. The simplest and most theoretically sound scenario of dark energy is the vacuum energy (ΛCDM) with a constant equation of state (EoS) $w_{\rm x} = p_{\rm x}/\rho_{\rm x} = -1$ where $w_{\rm x}$ denotes the EoS of dark energy. This model is consistent with most of the current astronomical observations, but suffers from the cosmological constant problem [6] and age problem [7] as well. Recently, Hubble tension may also provide evidences for physics beyond ΛCDM [8].

The general approach to studying dark energy is to assume either a theoretical model or an EoS, and then use observational data to limit relevant parameters, see for example, for spatially-flat ΛCDM the Hubble constant and the matter density parameter are constrained as: $H_0 = (67.4 \pm 0.5)$ km s^{-1}Mpc^{-1}, $\Omega_{\rm m} = 0.315 \pm 0.007$, respectively; while for EoS ($w_{\rm x} = w_0 + \frac{w_a z}{1+z}$) parameterized model, the related parameters are limited as: $H_0 = (68.31 \pm 0.82)$ km s^{-1}Mpc^{-1}, $w_0 = -0.957 \pm 0.080$, and $w_a = -0.29^{+0.32}_{-0.26}$ [5]. Statistical methods, such as the maximum likelihood [7, 9–12], are generally used to analyze the observational data to fit the parameters. These statistical method yields the best statistical results, but it is easy to eliminate some interesting (possibly important) data. Here we propose a model-independent method by using the Lagrange mean value theorem to analyze $H(z)$ parameter data. We find that the EoS of dark energy may be less than -1 at some redshifts and the accelerated phase may occur earlier than we previously thought.

The paper is organized as follows. In the next Section, we will present $H(z)$ parameter data and derive the equations needed to analyze these data. In Sec. III, We will provide the data and results obtained from the analysis. Finally, we will briefly summarize and discuss our results in section IV.

II. THEORETICAL METHOD AND $H(z)$ PARAMETER DATA

In this Section, we will present 63 $H(z)$ parameter data obtained recently, then introduce the Lagrange mean value theorem and combine it with Friedmann equations to derive equations needed to analyze $H(z)$ parameter data.

* Corresponding author
† Electronic address: yangrongjia@tsinghua.org.cn

A. Theoretical method

Assuming a Friedmann-Robertson-Walker-Lemaître (FRWL) spacetime

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1}$$

where $a(t)$ is the scale factor, K denotes the curvature of the space with $K = +1$, 0, and -1 corresponding to a closed, flat and open universe, respectively. We use the unit $c = 1$ here. According to the Planck 2018 results, the spacetime is spatially flat: $\Omega_{\mathrm{K0}} = 0.001 \pm 0.002$ [5]. So we consider a spatially flat FRWL spacetime here, the Friedmann equations take the form

$$H^2 = \frac{8\pi G}{3}\rho, \tag{2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right), \tag{3}$$

or equivalently

$$\dot{H} = -4\pi G(\rho + p), \tag{4}$$

where the $H \equiv \dot{a}/a$ is the Hubble parameter with the dot denoting the derivative with respect to the cosmic time t. The total energy density ρ and pressure p contain contributions coming from the radiation, nonrelativistic matter, and other components. Because $dz = -(1+z)Hdt$, we have

$$\dot{H} = -(1+z)H\frac{dH}{dz}. \tag{5}$$

Combining Eqs. (4) and (5), yields

$$\frac{dH}{dz} = \frac{4\pi G}{(1+z)H}(\rho + p) = \frac{4\pi G\rho(1+w_{\mathrm{t}})}{(1+z)H}, \tag{6}$$

where w_{t} is the total EoS. From this equation, we can judge whether the total EoS is greater than, equal to, or less than -1: see for example, if $dH/dz < 0$, we have $w_{\mathrm{x}} \leq w_{\mathrm{t}} \leq -1$ because of the positive of H and ρ. In an era dominated by dark energy, we can also determine with Eq. (6) wether the EoS of dark energy is equal to -1: if $dH/dz = 0$, then one have $w_{\mathrm{x}} \simeq w_{\mathrm{t}} = -1$. If $dH/dz \leq 0$, we know the Universe is experiencing an accelerated expansion. But if $dH/dz > 0$, we can't judge whether the Universe speeds up. At this point, we need another important physical quantity, the deceleration parameter, which is defined as

$$q = -1 + (1+z)\frac{1}{H}\frac{dH}{dz}. \tag{7}$$

Now a question naturally rise: if we have some $H(z)$ parameter data, how can we use them to directly determine dH/dz or q? Think of Lagrange mean value theorem in Calculus, which states: for a continuous and differentiable function $f(x)$, there exists $x_1 < x_{12} < x_2$ satisfying

$$\frac{df}{dx}\Big|_{x=x_{12}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{8}$$

Applying this theorem to Hubble parameter which we assume is a continuous and differentiable function of z, and taking function $H(z)$ as $f(x)$ in (8), we have

$$H'(z_{ij}) \equiv \frac{dH}{dz}\Big|_{z=z_{ij}} = \frac{H(z_i) - H(z_j)}{z_i - z_j}, \tag{9}$$

where $z_j < z_{ij} < z_i$. If $z_i - z_j \ll 1$, $H'(z_{ij})$ will be large in general, which will make relevant results less credible. Since z_{ij} and $H(z_{ij})$ in Eq. (7) are unknown, we take approximatively: $z_{ij} = (z_i + z_j)/2$ and $H(z_{ij}) \simeq [H(z_i) + H(z_j)]/2$, which can be called as mid-value approximate method. Then we have

$$q(z_{ij}) \simeq -1 + \frac{(2 + z_i + z_j)\,H'(z_{ij})}{H(z_i) + H(z_j)}. \tag{10}$$

index	z	$H(z)$[km s^{-1}Mpc^{-1}]	σ_H	Reference	Method	index	z	$H(z)$	σ_H	Reference	Method
z_1	0	74.03	1.42	[8]	SN Ia/Cepheid	z_{33}	0.51	90.4	1.9	[13]	Clustering
z_2	0.07	69	19.6	[14]	DA	z_{34}	0.52	94.35	2.65	[15]	Clustering
z_3	0.1	69	12	[16]	DA	z_{35}	0.56	93.33	2.32	[15]	Clustering
z_4	0.12	68.6	26.2	[14]	DA	z_{36}	0.57	92.9	7.8	[17]	Clustering
z_5	0.17	83	8	[16]	DA	z_{37}	0.59	98.48	3.19	[15]	Clustering
z_6	0.1797	75	4	[18]	DA	z_{38}	0.5929	104	13	[18]	DA
z_7	0.1993	75	5	[18]	DA	z_{39}	0.6	87.9	6.1	[19]	Clustering
z_8	0.2	72.9	29.6	[14]	DA	z_{40}	0.61	97.3	2.1	[13]	Clustering
z_9	0.24	79.69	2.65	[20]	Clustering	z_{41}	0.64	98.82	2.99	[15]	Clustering
z_{10}	0.27	77	14	[16]	DA	z_{42}	0.6797	92	8	[18]	DA
z_{11}	0.28	88.8	36.6	[14]	DA	z_{43}	0.73	97.3	7	[19]	Clustering
z_{12}	0.3	81.7	6.22	[21]	Clustering	z_{44}	0.75	98.8	33.6	[22]	Clustering
z_{13}	0.31	78.17	4.74	[15]	Clustering	z_{45}	0.7812	105	12	[18]	DA
z_{14}	0.34	83.8	3.66	[20]	Clustering	z_{46}	0.8754	125	17	[18]	DA
z_{15}	0.35	82.7	8.4	[23]	Clustering	z_{47}	0.88	90	40	[16]	DA
z_{16}	0.3519	83	14	[18]	DA	z_{48}	0.9	117	23	[16]	DA
z_{17}	0.36	79.93	3.39	[15]	Clustering	z_{49}	0.978	113.72	14.63	[24]	Clustering
z_{18}	0.38	81.5	1.9	[13]	Clustering	z_{50}	1.037	154	20	[18]	DA
z_{19}	0.3802	83	13.5	[25]	DA	z_{51}	1.23	131.44	12.42	[24]	Clustering
z_{20}	0.40	82.04	2.03	[15]	DA	z_{52}	1.3	168	17	[16]	DA
z_{21}	0.4	95	17	[16]	DA	z_{53}	1.363	160	33.6	[26]	DA
z_{22}	0.4004	77	10.2	[25]	DA	z_{54}	1.43	177	18	[16]	DA
z_{23}	0.4247	87.1	11.2	[25]	DA	z_{55}	1.526	148.11	12.71	[24]	Clustering
z_{24}	0.4293	91.8	5.3	[25]	DA	z_{56}	1.53	140	14	[16]	DA
z_{25}	0.43	86.45	3.68	[20]	Clustering	z_{57}	1.75	202	40	[16]	DA
z_{26}	0.44	82.6	7.8	[19]	Clustering	z_{58}	1.944	172.63	14.79	[24]	Clustering
z_{27}	0.44	84.81	1.83	[15]	Clustering	z_{59}	1.965	186.5	50.4	[26]	DA
z_{28}	0.4497	92.8	12.9	[25]	DA	z_{60}	2.3	224	8	[27]	Clustering
z_{29}	0.47	89	34	[28]	DA	z_{61}	2.33	224	8	[29]	Clustering
z_{30}	0.4783	80.9	9	[25]	DA	z_{62}	2.34	222	7	[30]	Clustering
z_{31}	0.48	87.79	2.03	[15]	DA	z_{63}	2.36	226	8	[31]	Clustering
z_{32}	0.48	97	62	[16]	DA						

TABLE I: Hubble parameter compilation from cosmic chronometers (DA) or from the radial BAO surveys (clustering).

If $z_i - z_j$ is large, in general, Eq. (10) will be not valid. Taking the uncertainty on the values of $H(z)$ into account, the uncertainties associated to the $H'(z)$ and $q(z)$ are given by, respectively

$$\sigma_{H'} = \frac{\sqrt{\sigma_{H_i}^2 + \sigma_{H_j}^2}}{z_i - z_j}, \tag{11}$$

and

$$\sigma_q = \frac{2(2 + z_i + z_j)H_i}{(H_i + H_j)^2} \frac{\sqrt{\sigma_{H_i}^2 + \sigma_{H_j}^2}}{z_i - z_j}. \tag{12}$$

With $\sigma_{H'}$ and σ_q, we can determine whether the results are credible at 1 σ confidence level.

index	$H'(z)$	$\sigma_{H'}$	$q(z)$	σ_q	index	$H'(z)$	$\sigma_{H'}$	$q(z)$	σ_q
$z_{61} \in (0, 0.1797)$	5.398	23.62	-0.921	0.348	$z_{2714} \in (0.34, 0.44)$	10.1	40.92	-0.833	0.679
$z_{71} \in (0, 0.1993)$	4.867	26.08	-0.928	0.387	$z_{359} \in (0.24, 0.56)$	42.625	11.006	-0.31	0.192
$z_{91} \in (0, 0.24)$	23.583	12.527	-0.656	0.189	$z_{409} \in (0.24, 0.61)$	47.595	9.138	-0.233	0.162
$z_{131} \in (0, 0.31)$	13.355	15.962	-0.797	0.249	$z_{419} \in (0.24, 0.64)$	47.825	9.988	-0.228	0.178
$z_{141} \in (0, 0.34)$	28.745	11.547	-0.574	0.182	$z_{3418} \in (0.38, 0.52)$	91.786	23.291	0.514	0.412
$z_{171} \in (0, 0.36)$	16.389	10.209	-0.749	0.162	$z_{3420} \in (0.40, 0.52)$	102.583	27.818	0.698	0.493
$z_{181} \in (0, 0.38)$	19.658	6.242	-0.699	0.1	$z_{3718} \in (0.38, 0.59)$	80.857	17.68	0.334	0.319
$z_{201} \in (0, 0.4)$	20.025	6.193	-0.692	0.1	$z_{3431} \in (0.48, 0.52)$	164.00	83.454	1.701	1.424
$z_{251} \in (0.0, 0.43)$	28.884	9.173	-0.562	0.15	$z_{3727} \in (0.44, 0.59)$	91.133	24.518	0.507	0.436
$z_{271} \in (0.0, 0.44)$	24.5	5.264	-0.624	0.086	$z_{3731} \in (0.48, 0.59)$	97.182	34.374	0.602	0.599
$z_{311} \in (0.0, 0.48)$	28.667	5.161	-0.561	0.086	$z_{4324} \in (0.4293, 0.73)$	18.291	18.959	-0.694	0.326
$z_{139} \in (0.24, 0.31)$	-21.714	77.578	-1.351	1.241	$z_{5150} \in (1.037, 1.23)$	-116.891	121.983	-2.747	1.679
$z_{179} \in (0.24, 0.36)$	2.00	35.857	-0.967	0.585	$z_{5550} \in (1.037, 1.526)$	-12.045	48.46	-1.182	0.718
$z_{189} \in (0.24, 0.38)$	12.929	23.291	-0.79	0.383	$z_{5650} \in (1.037, 1.53)$	-28.398	49.519	-1.441	0.733
$z_{209} \in (0.24, 0.40)$	14.688	20.864	-0.76	0.346	$z_{5552} \in (1.3, 1.526)$	-88.009	93.92	-2.344	1.344
$z_{259} \in (0.24, 0.43)$	35.579	23.868	-0.428	0.399	$z_{5652} \in (1.3, 1.53)$	-121.739	95.75	-2.909	1.365
$z_{279} \in (0.24, 0.44)$	25.60	16.102	-0.583	0.271	$z_{5554} \in (1.43, 1.526)$	-300.938	229.532	-5.588	3.188
$z_{319} \in (0.24, 0.48)$	33.75	13.9091	-0.452	0.237	$z_{5654} \in (1.43, 1.53)$	-370	228.035	-6.789	3.152
$z_{339} \in (0.24, 0.51)$	39.667	12.077	-0.359	0.208	$z_{5854} \in (1.43, 1.944)$	-8.502	45.324	-1.131	0.688
$z_{2014} \in (0.34, 0.40)$	-29.333	69.755	-1.485	1.14					

TABLE II: $H'(z)$ and $q(z)$ data obtained from $H(z)$ parameter data.

B. $H(z)$ parameter data

The data set we use consists of 1 $H(z)$ measurement from SNIa observation, 34 $H(z)$ measurements obtained by calculating the differential ages of galaxies, which is called cosmic chronometer, and 28 $H(z)$ measurements inferred from the baryon acoustic oscillation (BAO) peak in the galaxy power spectrum, as listed in Table I. In three cases, the datasets are given with their 1σ confidence interval.

III. APPLICATIONS

In this Section, we apply Eqs. (9), (10), (11), and (12) to investigate the evolution of the Universe with the observational Hubble parameter data. In order to avoid significant errors, we have considered the following limitations in the process of analyzing the $H(z)$ data: $0.1 \lesssim z_i - z_j \lesssim 0.5$, $\sigma_H \leq 5$ if $H \leq 100$, and $\sigma_H \leq 20$ if $H \geq 100$. The data for $H'(z)$, $q(z)$ at 1 σ confidence level are listed in Table II, from which we can conclude:

(a) At redshifts z_{61}, z_{71}, z_{91}, z_{131}, z_{141}, z_{171}, z_{181}, z_{201}, z_{251}, z_{271}, z_{311}, z_{139}, z_{179}, z_{189}, z_{209}, z_{259}, z_{279}, z_{319}, z_{339}, z_{2014}, z_{2714}, z_{359}, z_{409}, z_{419}, z_{4324}, z_{5150}, z_{5550}, z_{5650}, z_{5552}, z_{5652} z_{5554}, z_{5654}, and z_{5854}, the Universe experiences an accelerated expansion at 1 σ confidence level.

(b) At redshifts z_{3418}, z_{3420}, z_{3718}, z_{3431}, z_{3727}, and z_{3731}, the Universe experiences an decelerated expansion at 1 σ confidence level.

(c) At redshifts z_{139}, z_{2014}, z_{5150}, z_{5550}, z_{5650}, z_{5552}, and z_{5854}, since $H'(z) < 0$, implying $w_{\rm x} \leq w_{\rm t} < -1$, but not at at 1 σ confidence level. However, we can infer that $w_{\rm x} \leq w_{\rm t} < -1$ at redshifts z_{5652}, z_{5554}, and z_{5654} at at 1 σ confidence level.

According to the Planck 2018 results [5], the matter density parameter for the spatially-flat ΛCDM was constrained as: $\Omega_{\rm m} = 0.315 \pm 0.007$, implying that the phase transition from deceleration to acceleration of the Universe occurs at the redshift $z \simeq 0.632$. Result (b), however, shows that there exists at least one decelerated phase in the redshift range $(0.38, 0.59)$ at 1 σ confidence level. In addition, results (a) and (c) also indicate that the EoS of dark energy may be less than -1 at some redshifts and the accelerated phase may occur earlier than we previously thought.

IV. CONCLUSIONS AND DISCUSSIONS

Using the Lagrange mean value theorem in Calculus, we have analysed $H(z)$ parameter data with some conditions and find that: (1) there exists at least one decelerated phase at 1 σ confidence level in the redshift range $(0.38, 0.59)$; (2) the EoS of dark energy may be less than -1 at 1 σ confidence level at some redshifts in the redshift range $(1.3, 1.53)$; (3) there exist at least one accelerated phase at 1 σ confidence level in the redshift range $(1.037, 1.944)$. The $q(z)$ data we obtained can be used to investigate cosmological models. The results may provide clues to address Hubble tension and possible evidences for physics beyond ΛCDM.

Acknowledgments

We thank Shina Tan, Yuan Liu and Tiefu Zhang for helpful discussions. This work is supported in part by Hebei Provincial Natural Science Foundation of China (Grant No. A2021201034).

---

[1] A. G. Riess *et al.*, "Observational evidence from supernovae for an accelerating universe and a cosmological constant," *Astron. J.*, vol. 116, pp. 1009–1038, 1998.
[2] S. Perlmutter *et al.*, "Measurements of Omega and Lambda from 42 high redshift supernovae," *Astrophys. J.*, vol. 517, pp. 565–586, 1999.
[3] M. Tegmark *et al.*, "Cosmological parameters from SDSS and WMAP," *Phys. Rev.*, vol. D69, p. 103501, 2004.
[4] G. Hinshaw *et al.*, "Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results," *Astrophys. J. Suppl.*, vol. 208, p. 19, 2013.
[5] N. Aghanim *et al.*, "Planck 2018 results. VI. Cosmological parameters," *Astron. Astrophys.*, vol. 641, p. A6, 2020. [Erratum: Astron.Astrophys. 652, C4 (2021)].
[6] S. M. Carroll, "The Cosmological constant," *Living Rev.Rel.*, vol. 4, p. 1, 2001.
[7] R.-J. Yang and S. N. Zhang, "The age problem in ΛCDM model," *Mon.Not.Roy.Astron.Soc.*, vol. 407, pp. 1835–1841, 2010.
[8] A. G. Riess, S. Casertano, W. Yuan, L. M. Macri, and D. Scolnic, "Large Magellanic Cloud Cepheid Standards Provide a 1% Foundation for the Determination of the Hubble Constant and Stronger Evidence for Physics beyond ΛCDM," *Astrophys. J.*, vol. 876, no. 1, p. 85, 2019.
[9] R. J. Yang, S. N. Zhang, and Y. Liu, "Constraints on the generalized tachyon field models from latest observational data," *JCAP*, vol. 01, p. 017, 2008.
[10] R. Yang, B. Chen, H. Zhao, J. Li, and Y. Liu, "Test of conformal gravity with astrophysical observations," *Phys. Lett. B*, vol. 727, pp. 43–47, 2013.
[11] S. Nesseris and L. Perivolaropoulos, "Comparison of the legacy and gold snia dataset constraints on dark energy models," *Phys. Rev. D*, vol. 72, p. 123519, 2005.
[12] R. Lazkoz, S. Nesseris, and L. Perivolaropoulos, "Exploring Cosmological Expansion Parametrizations with the Gold SnIa Dataset," *JCAP*, vol. 11, p. 010, 2005.
[13] S. Alam *et al.*, "The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: cosmological analysis of the DR12 galaxy sample," *Mon. Not. Roy. Astron. Soc.*, vol. 470, no. 3, pp. 2617–2652, 2017.
[14] C. Zhang, H. Zhang, S. Yuan, T.-J. Zhang, and Y.-C. Sun, "Four new observational $H(z)$ data from luminous red galaxies in the Sloan Digital Sky Survey data release seven," *Res. Astron. Astrophys.*, vol. 14, no. 10, pp. 1221–1233, 2014.
[15] Y. Wang *et al.*, "The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: tomographic BAO analysis of DR12 combined sample in configuration space," *Mon. Not. Roy. Astron. Soc.*, vol. 469, no. 3, pp. 3762–3774, 2017.
[16] D. Stern, R. Jimenez, L. Verde, M. Kamionkowski, and S. Stanford, "Cosmic Chronometers: Constraining the Equation of State of Dark Energy. I: H(z) Measurements," *JCAP*, vol. 02, p. 008, 2010.
[17] L. Anderson *et al.*, "The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: measuring D_A and H at z = 0.57 from the baryon acoustic peak in the Data Release 9 spectroscopic Galaxy sample," *Mon. Not. Roy. Astron. Soc.*, vol. 439, no. 1, pp. 83–101, 2014.
[18] M. Moresco *et al.*, "Improved constraints on the expansion rate of the Universe up to z˜1.1 from the spectroscopic evolution of cosmic chronometers," *JCAP*, vol. 08, p. 006, 2012.
[19] C. Blake *et al.*, "The WiggleZ Dark Energy Survey: Joint measurements of the expansion and growth history at z < 1," *Mon. Not. Roy. Astron. Soc.*, vol. 425, pp. 405–414, 2012.
[20] E. Gaztanaga, A. Cabre, and L. Hui, "Clustering of Luminous Red Galaxies IV: Baryon Acoustic Peak in the Line-of-Sight Direction and a Direct Measurement of H(z)," *Mon. Not. Roy. Astron. Soc.*, vol. 399, pp. 1663–1680, 2009.
[21] A. Oka, S. Saito, T. Nishimichi, A. Taruya, and K. Yamamoto, "Simultaneous constraints on the growth of structure and cosmic expansion from the multipole power spectra of the SDSS DR7 LRG sample," *Mon. Not. Roy. Astron. Soc.*, vol. 439, pp. 2515–2530, 2014.

[22] N. Borghi, M. Moresco, and A. Cimatti, "Toward a Better Understanding of Cosmic Chronometers: A New Measurement of H(z) at z $\sim$ 0.7," *Astrophys. J. Lett.*, vol. 928, no. 1, p. L4, 2022.

[23] C.-H. Chuang and Y. Wang, "Modeling the Anisotropic Two-Point Galaxy Correlation Function on Small Scales and Improved Measurements of $H(z)$, $D_A(z)$, and $\beta(z)$ from the Sloan Digital Sky Survey DR7 Luminous Red Galaxies," *Mon. Not. Roy. Astron. Soc.*, vol. 435, pp. 255–262, 2013.

[24] G.-B. Zhao *et al.*, "The clustering of the SDSS-IV extended Baryon Oscillation Spectroscopic Survey DR14 quasar sample: a tomographic measurement of cosmic structure growth and expansion rate based on optimal redshift weights," *Mon. Not. Roy. Astron. Soc.*, vol. 482, no. 3, pp. 3497–3513, 2019.

[25] M. Moresco, L. Pozzetti, A. Cimatti, R. Jimenez, C. Maraston, L. Verde, D. Thomas, A. Citro, R. Tojeiro, and D. Wilkinson, "A 6% measurement of the Hubble parameter at $z \sim 0.45$: direct evidence of the epoch of cosmic re-acceleration," *JCAP*, vol. 05, p. 014, 2016.

[26] M. Moresco, "Raising the bar: new constraints on the Hubble parameter with cosmic chronometers at z $\sim$ 2," *Mon. Not. Roy. Astron. Soc.*, vol. 450, no. 1, pp. L16–L20, 2015.

[27] N. G. Busca *et al.*, "Baryon Acoustic Oscillations in the Ly-α forest of BOSS quasars," *Astron. Astrophys.*, vol. 552, p. A96, 2013.

[28] A. Ratsimbazafy, S. Loubser, S. Crawford, C. Cress, B. Bassett, R. Nichol, and P. Väisänen, "Age-dating Luminous Red Galaxies observed with the Southern African Large Telescope," *Mon. Not. Roy. Astron. Soc.*, vol. 467, no. 3, pp. 3239–3254, 2017.

[29] J. E. Bautista *et al.*, "Measurement of baryon acoustic oscillation correlations at $z = 2.3$ with SDSS DR12 Lyα-Forests," *Astron. Astrophys.*, vol. 603, p. A12, 2017.

[30] T. Delubac *et al.*, "Baryon acoustic oscillations in the Lyα forest of BOSS DR11 quasars," *Astron. Astrophys.*, vol. 574, p. A59, 2015.

[31] A. Font-Ribera *et al.*, "Quasar-Lyman α Forest Cross-Correlation from BOSS DR11 : Baryon Acoustic Oscillations," *JCAP*, vol. 05, p. 027, 2014.